Spectral Methods for Incompressible Navier Stokes Equations

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CSS 555: Numerical Methods for PDEs

Abstract

We solved the incompressible unsteady Navier Stokes (UNS) equations using the spectral element method. We implemented the Orszag-Kells algorithm to solve the UNS equations on a rectangular cavity with various boundary conditions(BCs). The code was validated against the standard steady Lid-Driven cavity problem. Other BCs such as transient Dirichlet BC (oscillatory LDC) and Neumann BC (flow between two parallel plates) were then implemented.

Problem description

The unsteady Navier Stokes (UNS) equations govern the dynamics of (incompressible) fluid flow. The dimensionless form is given below:

$$\nabla.\mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}$$

where Re is the Reynolds number. We solve the above equations on a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$

Numerical method

To solve the UNS equations, we use spectral element method with one element having $P_N - P_{N-2}$ basis for velocity-pressure on GLL-GL nodes with BDF3/EXT3 as time stepper for temporal evolution of the system. We implement the **Orszag-Kells** algorithm which decouples the non-linear system of equations as follows. The first step involves calculating an intermediate velocity field $\tilde{u}$:

$$\tilde{\mathbf{u}} = -\frac{1}{\beta_0}\sum_{j=1}^{k}\beta_j \mathbf{u}^{n-j} + \frac{\Delta t}{\beta_0}\sum_{j=1}^{k}\alpha_j(\mathbf{f} - \mathbf{u}.\nabla \mathbf{u})^{n-j}$$

The second step called the pressure projection step projects $\tilde{u}$ onto a divergence free space $\tilde{\tilde{u}}$ using a decoupled pressure Poisson equation solution by neglecting the viscous term.

$$\nabla^2\phi = \nabla.\tilde{\mathbf{u}}, \quad \nabla\phi.\hat{\mathbf{n}} = \tilde{\mathbf{u}}.\hat{\mathbf{n}}$$

$$\tilde{\tilde{\mathbf{u}}} = \tilde{\mathbf{u}} - \nabla\phi, \quad p^n = \frac{\beta_0}{\Delta t}\phi$$

Numerical Method (contd.)

The final step involves solving the Helmholtz equation involving the viscous terms and enforcing BCs.

$$\mathbf{u}^n - \frac{\Delta t}{\beta_0 Re}\nabla^2 \mathbf{u}^n = \tilde{\tilde{\mathbf{u}}}$$

The entire algorithm is optimized for speed using Fast Diagonalization method and the advection component is over-integrated to ensure stability at high Re.

Results: Steady Dirichlet BC

The LDC problem involves flow in a cavity ($L_x = L_y = 1$) with the top lid moving with a constant velocity. This flow at Re $= 400$ gives rise to interesting flow physics with a major vortex near the center of the cavity and the secondary vortices near the bottom wall.

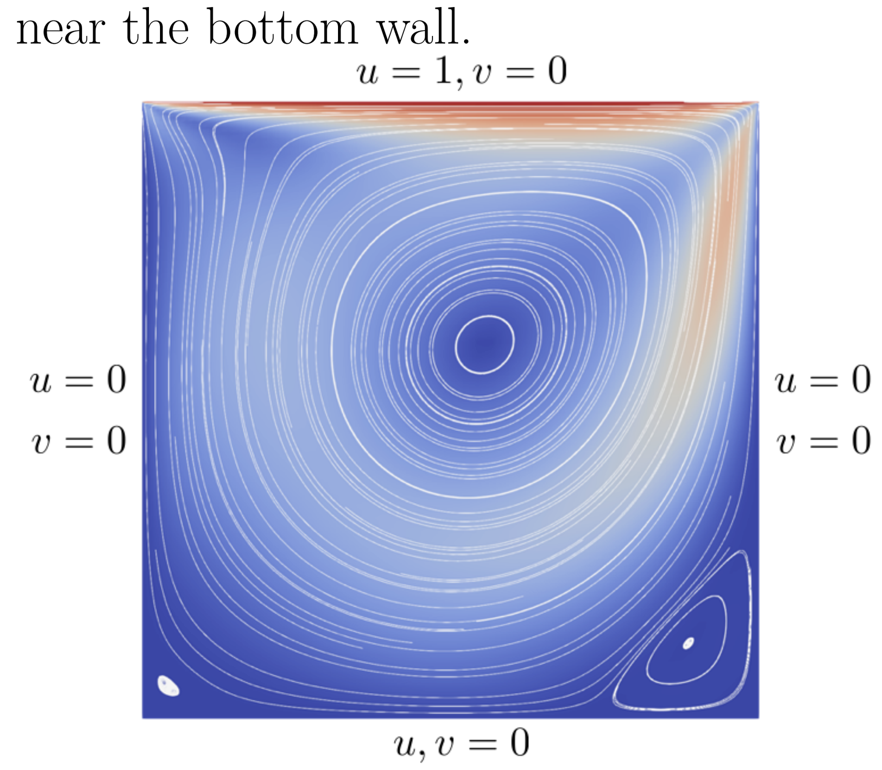

Figure 1: Contours of velocity magnitude for Re=400

Validation : LDC

We validated the code against the LDC problem. The results were compared against the data by Ghia *et.al.*. We see a good match for Re $= 100$ and 400.

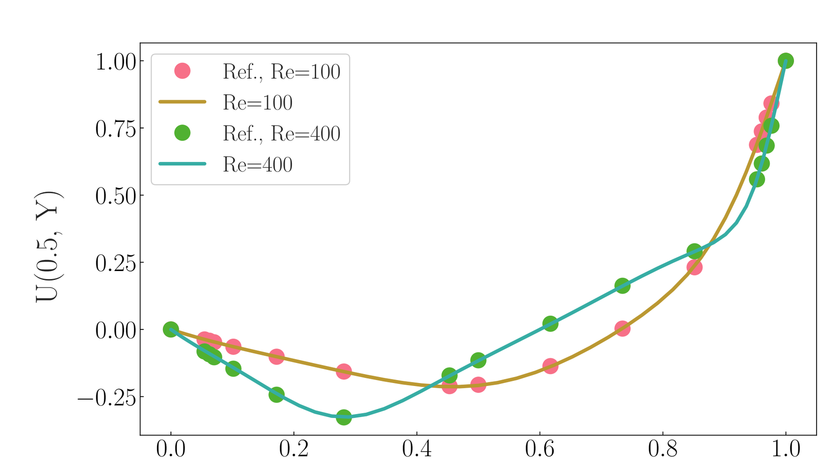

Figure 2: Validation of center line horizontal velocities at various Re

Convergence : LDC

To test the temporal and spatial convergence, we use the kinetic energy in the cavity as the parameter to test convergence against the finest solution (p=100, $\Delta t = 2.5\times10^{-4}$).

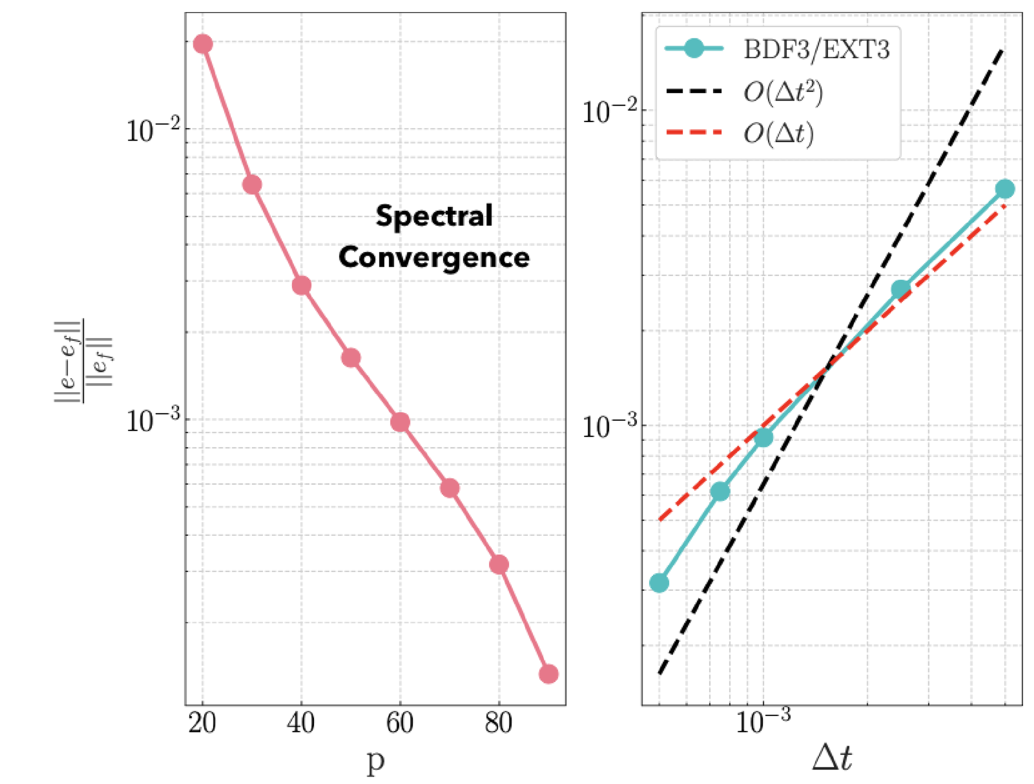

Figure 3: Spatial and Temporal convergence ($Re = 100$)

Results: Transient Dirichlet BC

In order to use a transient Dirichlet BC, we modified the velocity of the top lid to have the form $u(t) = u_{max}\cos(\omega t)$. This introduces another non-dimensional number called the Stokes number ($St = \omega u_{max}^2/\nu$). For this BC, we see a time periodic solution with time period equal to that of the oscillation (T). Further, the flow is symmetric (about the axis perpendicular to the transient boundary) for times t and $t + 0.5T$.

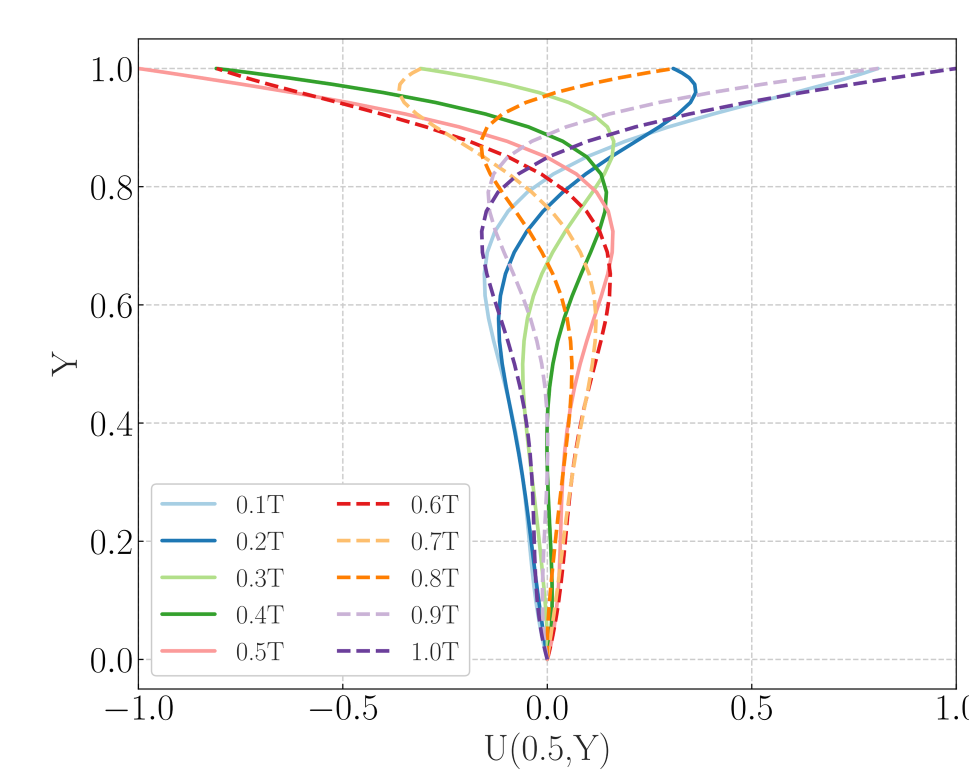

Figure 4: Center line horizontal velocities for Re $= 100$ and St $= 105$

Results: Neumann BC

To implement the Neumann BC, we studied the flow between two parallel plates. We initialized the left boundary with a uniform velocity and gave outflow conditions (Neumann) at the right boundary. Since flow between plates require a pressure gradient, a homogeneous Dirichlet reference pressure was applied at the right boundary.

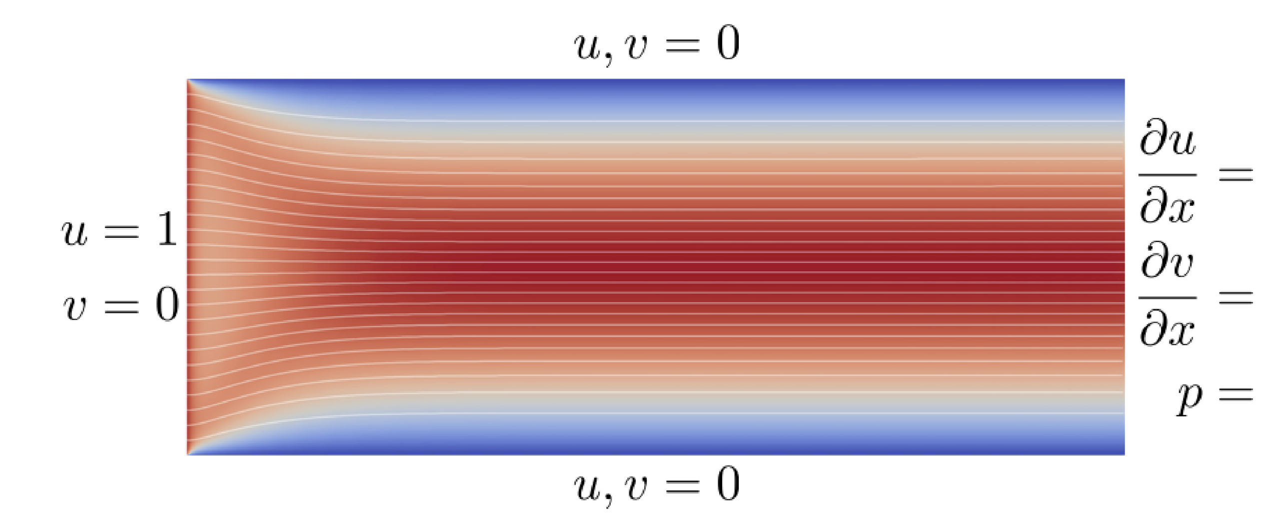

Figure 5: Horizontal velocity contours for Re $= 10$ with Lx $= 2.5$, Ly $= 1.0$

From the contours we can see that the boundary layers start developing on both the walls and eventually reach a "fully-developed" state. The line plots below show the radial velocity profiles at various stages of the developing flow. We see that the flow is fully developed at $X = 0.3$. Further we see that the fully developed profile is parabolic and is symmetric about the center which matches the theoretical predictions.

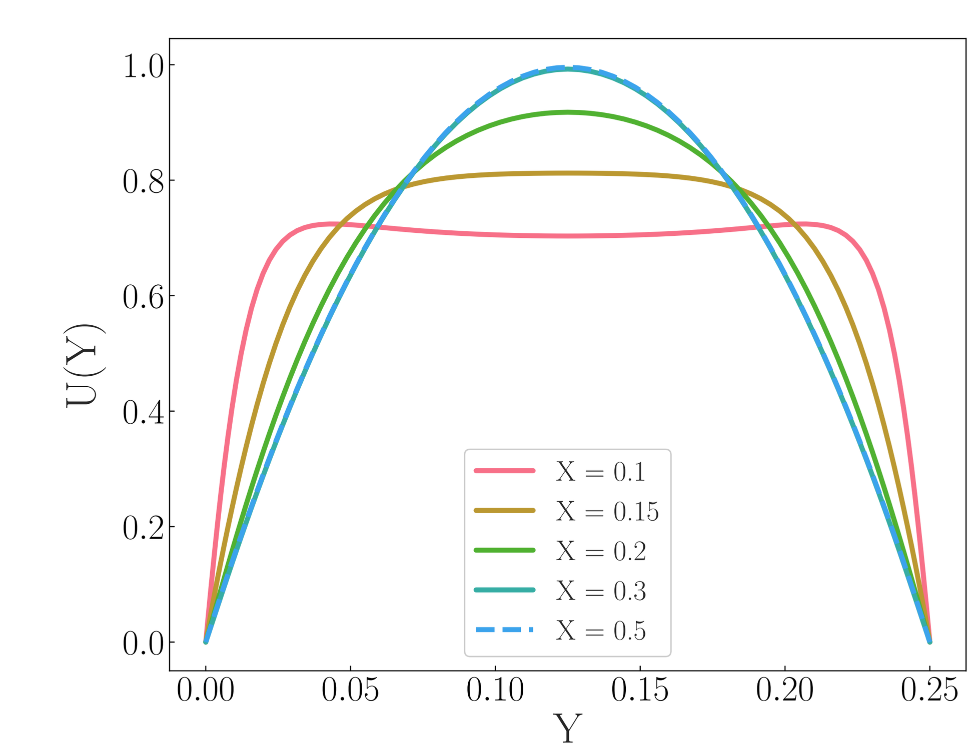

Figure 6: Horizontal Velocity profiles at various stages of developing flow for Re $= 10$